Counting in Two Ways[*]

Ray Li (`rayyli@stanford.edu`)

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1 Problems

1. In a certain committee, each member belongs to exactly three subcommittees, and each subcommittee has exactly three members. Prove that the number of members equals to the number of subcommittees.

2. (Putnam 1965) In a tennis tournament, every player plays every other player. Suppose that player i won w_i of his games and lost ℓ_i of them. Prove $\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} \ell_i^2$.

3. (IMC 2002) Two hundred students participated in a mathematical contest. They had 6 problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

4. (IMO 1998) In a competition, there are a contestants and b judges, where $b \geq 3$ is odd. Each judge rates each contestant as either "pass" or "fail". Suppose k is such that, for any two judges, their ratings coincide for at most k contestants. Prove that $\frac{k}{a} \geq \frac{b-1}{2b}$.

5. (Russia 1996) In the Duma there are 1600 delegates, who have formed 16,000 committees of 80 people each. Prove that one can find two committees having no fewer than four common members.

6. (Hamming Bound) Let $C \subseteq \{0,1\}^n$ be a set of binary strings of length n and d be a positive odd integer. Suppose that for any two distinct strings $c, c' \in C$, we have that $c_i \neq c'_i$ for at least d values of i. If $t = (d-1)/2$, prove that

$$|C| \leq \frac{2^n}{\sum_{k=0}^{t} \binom{n}{k}}.^{[1]}$$

7. (China TST 1992) Sixteen students took part in a math competition where every problem was a multiple choice question with four choices. Any two students had at most one answer in common. Determine the maximum number of questions.

[1]This gives an upper bound on the size of *error correcting codes*, C, that code against δn bit flips. Note that it is possible to recover the any string of C that is corrupted by δn bit flips. Using $\binom{n}{pn} \approx 2^{H(p)n}$, where $H(p) :=$ $-(p \log p + (1-p) \log(1-p))$ is the binary entropy function, we have $|C| \lesssim 2^{n(1-H(\delta))}$.

8. (China 1993) A group of 10 people went to a bookstore. It is known that (1) Everyone bought exactly 3 books; (2) For every two persons, there is at least one book that both of them bought. What is the least number of people that could have bought the book purchased by the greatest number of people?

9. (USAMO 2011) Let A be a set with $|A| = 225$, meaning that A has 225 elements. Suppose further that there are eleven subsets A_1, ..., A_{11} of A such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup A_2 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which equality holds.

10. (APMO 2006) In a circus, there are n clowns who dress and paint themselves up using a selection of 12 distinct colors. Each clown is required to use at least five different colors. One day, the ringmaster of the circus orders that no two clowns have exactly the same set of colors and no more than 20 clowns may use any one particular color. Find the largest number n of clowns so as to make the ringmasters order possible.

11. (IMO Shortlist 2010) $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and l_i be respectively the number of wins and losses of the i-th player. Prove that $\sum_{i=1}^{n} (w_i - l_i)^3 \geq 0$.

12. (IMO 1987) Let n and k be positive integers and let S be a set of n points in the plane such that: (1) no three points in S are collinear, and (2) for every P in S, there are at least k points in S equidistant from P. Prove that $k < \frac{1}{2} + \sqrt{2n}$.

13. (Iran 2010) There are n points in the plane such that no three of them are collinear. Prove that the number of triangles whose area is 1 and whose vertices are chosen from these n points is not greater than $\frac{2}{3}(n^2 - n)$.

14. (Iran 1999) Suppose that $C_1, \ldots, C_n$ ($n \geq 2$) are circles of radius one in the plane such that no two of them are tangent, and the subset of the plane formed by the union of these circles is connected. Let S be the set of points that belong to at least two circles. Show that $|S| \geq n$.

15. (RMM 2012) Given a finite group of boys and girls, a covering set of boys is a set of boys such that every girl knows at least one boy in that set; and a covering set of girls is a set of girls such that every boy knows at least one girl in that set. Prove that the number of covering sets of boys and the number of covering sets of girls have the same parity. (Acquaintance is assumed to be mutual.)

16. (IMO 2005) In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.